APPRAISAL OF RENE DESCARTES RATIONALISM AS A MODE OF DEVELOPING CRITICAL THINKING SKILLS FOR LEARNING MATHEMATICS IN NIGERIA'S SECONDARY SCHOOLS

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Abstract

Mathematics is accepted as a science subject concerned with critical reasoning but majority of students fail it in their examinations. This is the reason for appraising rationalism of Rene Descartes as an important mode for developing critical thinking skills for learning mathematics in secondary school. This study will also aid to improve the interest and poor performance of secondary school students in mathematics. In this way, their grades in external Examinations such as West African Senior School Certificate Examination will be high. The method adopted for this study is expository and hermeneutical design of critical thinking and mathematical concepts. The Chicago style was used in the citations. This study intends to give an explanatory framework on the concept of critical thinking in learning mathematics. More so, the work will indicate the practical application of the skill of critical thinking in learning mathematics. This research work covered learning of mathematics in secondary schools. The implication of this study is to significantly reduce failures in mathematics. It is ultimately urgent and important to reduce the massive failure of mathematics in secondary schools. Hence, the skills of critical thinking are accepted as a very essential step in every field of learning, and more so, in doing mathematics in the most recent decades. Therefore, this study draws insight from Descartes' who is a mathematician and rational thinker on the significance of critical thinking ability, particularly in learning mathematics.

1. Introduction

In the beginning of recorded history, mathematics discovery has been at the forefront of every civilized society and in use even in most primitive cultures and era. The needs for mathematics arose based on the want of society. The more complex human society, the more complex the mathematical needs. So, civilization developed from calculus to geometry which computes areas and volumes, angular measurements and other practical applications. The study of mathematics became a building block for civilization. In fact, it is considered to be crucial.

Mathematics is an important subject in the modern society especially in schools, workplace, business, and for personal decision making. It is fundamental to national development because mathematics provides tools for understanding sciences, engineering, technology and economics, (Kulbir, 2006).[1] Considering its importance, Nigerian government made mathematics, a compulsory subject in basic education level and secondary school as well as a prerequisite for admission into tertiary institutions.

Notwithstanding the huge importance placed on mathematics by the Nigerian society, it was observed that majority of the students lack interest in the subject and performs poorly in it, (Maduabum & Odili 2006, 64-68).[2] A good instance is the analysis of the West African Senior Secondary Certificate Examination (WASSCE) May/June from 2011 to 2015 shows that on the average, less than 50% of students passed mathematics at credit level. There has been other reasons for the poor performance in mathematics such as lack of interest, instructional factors, difficulty in understanding, teacher related factors and so on. Another study shows that the reason for the continued poor performance in mathematics could be attributed to students inability to critically think and analyze systematically the concepts of mathematics (Osarenren & Asiedu, 2007).[3] This paper agrees that lack of critical thinking is a major problem. Therefore, it confirms that critical thinking is an essential concept that is required to enhance performance in any subject especially in mathematical problems.

Critical thinking is a complex concept that involves cognitive skills and affective disposition. It is because most students do not possess these qualities they find it difficult to comprehend the concepts and perform badly in mathematics. Critical thinking involves the effort to collect, interpret, analyze and evaluate the information of a given purpose so as to arrive at reliable and valid conclusion. Hence, considering the importance of mathematics and its learning, critical thinking should be integrated and emphasized. It needs to be added in the curriculum for students to learn the skills and apply it so as to improve their performances in school. In this paper, Rene Descartes theories of rationalism offer the mode for developing the skills of critical reasoning. Descartes urged people to use hyperbolic doubt whenever assessing anything in life. His belief was that we, as thinking beings, should be skeptical when searching for any answers in life, and to not just accept information, firstly given to you. In other words, he opines that Critical thinking is

a system of thinking in which the thinker improves the quality of one's thinking by skillfully analyzing, evaluating, and reconstructing it.[4]

Having stated the above facts, this paper confirms that the development of critical thinking based on Rene Descartes words 'cogito ergo sum: I think, therefore I am', is the key to learning mathematics. It supposes self realization of the power of thinking capacity of a human person. This thinking power helps man in problem-solving. The ability to reason is central to logical thinking. It is a valuable skill which should be taught before and when mathematical processes are learnt. Students should learn how to form logical arguments and arrive at true conclusion within home and in school before and when learning mathematics. The capacity to engage extensively in argumentative discourse will sharpen the ability of children to conquer the fear for mathematics. It will help to comprehend it as a normal simple reasoning in daily life. In fact, when argumentative discussions are handled as a way of reasoning, it helps a great deal in understudying the complex mathematical statements. The learning of the processes of argumentative discourse involves complex patterns which includes the use of hypothetical statements with negative or counter factual consequences. In the course of learning argumentative constructions, children gain more complex cognitive processes especially when they are able to gain this exposure at home and in school. It will help them to experience complicated reasoning patterns. More so, the children will become more aware of thinking. It will develop quality of reasoning in the students and dispose them for learning mathematics. So far, the link between critical thinking and Mathematics is affirmed as stated that 'thus in some loose sense all of pure mathematics falls within the scope of logical thinking in the wider sense' (Hanna 2011, 196).[5] Therefore, the mastery of logical reasoning or thinking is the foundation for easier learning of mathematics.

Of course, to achieve all these, teachers training programme should include courses for critical thinking. In doing this, the teachers will be taught models of thinking strategies which will in-turn influence the students and enhance easier learning of the subject. When the teachers and students are deep in critical thinking skills, it will make each other to think, question issues, challenge ideas, generate solutions to problems and make intelligent decisions when faced with challenges (Semil 2006, 28-29).[6] This is because critical thinking involves deep reasoning and consideration of what one receives rather than outrightly accepting it as different idea (Mansoor and Pezeshki 2012, 153-160).[7] It is through critical thinking that problems are engaged in systematic and logical process

in order to find the truth. In learning mathematics, critical thinking would help students to understand the logical connection of ideas, construct and evaluate the problem, detect mistakes and systematically solve problems. And there are six cognitive skills that are central to critical thinking. They include interpretation, analysis, explanation, evaluation, self recognition and inference[8] which will be discussed later. Hence, concerning mathematics, critical thinking skills would enable a student to analyze and synthesize information to solve problems in broad range. The students will use the skills to collect, analyze, evaluate and conclude a mathematical problem through a given fact before them. Therefore, this paper draws insight from rationalism of Rene Descartes as a mode for developing critical thinking skills for learning mathematics in secondary schools.

2. Conceptual analysis

Historically, there have been strong links between mathematics and critical thinking since many great mathematicians were also great thinkers. In the ancient Greek origins, Pythagoras, Plato, Aristotle and Euclid were mathematicians as well as critical thinkers. The same can be said of many contemporary French, American, Asian, British, Russian and African mathematicians. It is important to state here that both critical thinking and mathematics are interwoven. However, critical or Logical thinking originated from classical Greek '*logos*' meaning word, thought, ideas, argument, account, reason or principle. Logical reasoning is process or principles in criteria of valid inference and demonstration. It is the study of how to critically think about propositions or statements that are either true or false. Logic explores and classifies the structure of statements and argument both through the study of formal system of inference and through the study of argument in natural language. Logic is the analysis of method of reasoning. It was Rene Descartes who popularized critical thinking through his philosophy of methodic doubt.

The following concepts are used interchangeably with critical thinking; Logic, critical reasoning, and logical reasoning. The Encyclopedia of Philosophy defined critical thinking as a careful goal-directed thinking.[9] Another definition states that critical thinking is the mode of thinking about any subject, content or problem in which the thinker improves the quality of his or her thinking by skillfully taking charge of the structure inherent in thinking and imposing intellectual standards

upon them, (Paul & Elder, 2010).[10] The central thing in logical thinking is the ability to reason. Reasoning is clearly a valuable skill which is often put to test when arguing or during argument. More then, Fisher and Scriven (1997) support that critical thinking is a "skilled and active interpretation and evaluation of observations, communication, information and argumentation.[11] Generally in philosophy, logic is understood as the method or framework for both philosophical reasoning and reasoning concerning any subject matter. Uduigwomen (2011) stated that logic is concerned with the principles of correct reasoning.[12] It has its own unique language and way of defining what is true or false. Logic is defined as the philosophical study of pure reasoning which deals with principles of correct argument. In our daily activities, arguments are formed in our statements and conclusion drawn.

Most times, statements and conclusions made are wrong. Logical reasoning is therefore, essential because it helps to point out how arguments should be made and how fallacies can be detected in an argument and avoided. More so, Uduigwomen, (2011) confirms that logical reasoning plays an important role both deductively and inductively. There are two forms of reasoning: deductive and inductive.[13]

Deductive Reasoning: this is the type of reasoning from general to particular. Here conclusions are drawn from general statements. It is always expressed in the form of syllogism. Thus, reasoning is prevalent in mathematics, sciences, religion and other fields of studies.

Inductive Reasoning: this is the reverse of deductive reasoning. It is applicable with empirical science. It deals with reasoning from particular to general laws. More so, it inferred conclusion from particular incidence. In other words, it takes specific examples as its premise and then draws a general conclusion from them, (Denley and Hall, 2016).[14] This form of reasoning gives space for various incidences of a particular specimen to be observed over a certain period of time. In mathematics also probability is the central concept of inductive logic, (Irving and Copi, 2001).[15] More so, logical reasoning in modern philosophical expressions is divided into two dimensions;

1. Analytic logic is used by analytic philosophers who emphasize the logical analysis of language to arrive at clear meaning of terms.

2. Symbolic logic is applied in mathematics where figures, symbols and alphabets are used to explain a phenomenon, example, **a+b=4, a=4-b.** There are two forms of symbolic logic, prepositional and predicate.

In other words, logical reasoning deals with all forms of arguments in which reasoning must take place within the logical framework defining the logical reasoning as formal science. The question should be, is reasoning something that we should be teaching? The fact remains that if logical thinking is all about reasoning, then there is no doubt it is the cornerstone of mathematical activities. A brief study of Rene Descartes life and philosophy as emphasized in his 'cogito ergo sum- I think therefore I exist' will help to understand this fact. Therefore, it is an undeniable supposition that logical reasoning is a science of correct reasoning necessary for solving problems especially in mathematical processes.

Mathematics is from Greek '*mathema*' meaning knowledge, study or learning. Mathematics is defined as the science of structure, relation and order that has evolved from elemental proactive of counting, measuring and describing the shapes of objects. It involves logical reasoning and quantitative calculation which has evolved into abstractions and various increasing degrees of idealization. In fact, it has developed beyond counting. In summary, the branches of mathematics include, quantity (number theory), structure (algebra), space (geometry), and change (mathematical analysis). It is from logic and abstractions that mathematics developed other forms like calculation, measurement, etc. Mathematics in the views of Watson and Mason (1998) relies on the structures of pure mathematics and mathematical thinking. Any problem in mathematics involves various forms of statements that are developed.[16] Any problem in mathematical lessons is the structure. It is known that almost all the mathematical systems are combinations of sets of axioms and theorem which is deduced from the axioms, (Jeremy, Gray, Craig et al., 2019).[17]

Mathematics is further defined as the study of topics such as structure, quantity, space, change and so on. It involves the use of techniques like abstraction and logical reasoning from counting, calculation, measurement and systematic study of the shapes and motion of physical object. The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greek philosophers such as Euclid, Plato and Aristotle, (Barnes 1995, 27).[18] Gradually, the analytical generality of predicate logic allowed the formalization of mathematics and drove the

investigation of set theory. It also allowed the development of Alfred Tarski's approach to model theory which provided the foundation of modern mathematical logic (Taski, 1948).[19]

Of course, logical reasoning is recognized as used in mathematics and obeys mathematical rules that can be made fully explicit, (Dries, 2016).[20] Some philosophers and logicians such as Russell, Zermelo, Hausdoffm, and Hilbert helped to discover and develop the first real theory in mathematical logic. In the early 20th century, there were strong impulse for developing mathematical logic and that obvious attempt provided a solid foundation for mathematics. Mathematical logic which is concerned with the relation of mathematics and logical reasoning has permeated other disciplines such as computer science and engineering. What differentiates mathematical logic from mathematics is the fact that statement about mathematical objects are taken seriously as mathematics objects in their own right. In mathematical logic, we formalize (formulate in a precise mathematical way) notions used informally by mathematicians such as:

1. **Property**
2. **Statement (in a given language)**
3. **Structure**
4. **Truth- what it means for a given statement to be true in a given structure**
5. **Proof (from a given set of axiom)**
6. **Algorithm**

In essence, mathematical logic is a sub branch of mathematics exploring the applications of formal reasoning to the subject of mathematics. Here, logical reasoning has connections to; mathematics, foundation of mathematics and theoretical computer science. It is divided into the following fields:

1. Set theory: this is the study of sets which are abstract collections of objects. There are two known statements in set theory which is axiom of choice and continuum hypothesis. Axiom of choice states that given a collection of non-empty set, there is a single set that contains exactly one element from each set in the collection. The set chooses one element from each set in the collection while the ability to make such choice is considered necessary by some since each set in the collection is non-empty. In solving the problems of set theory, 'Toposes' is used and it employs classical and non-classical logical reasoning.

2. Model theory: this theory studies the models of all formal theories. Theory is a set of formulas in a particular formal logical reasoning and signature while model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry. Hence, the methods of model theory focus on logical considerations than other fields. It is important to know that set of all models of a particular theory is called an elementary class. Therefore, classical model theory seeks to determine the properties of model in a particular elementary class. In other words, it can aid to determine whether certain classes of structures form elementary classes.
3. Recursion theory: this is also called computability theory. It studies the properties of computable correct functions and the turning degrees which divides incomputable functions into sets that have the same level of in-computability. It deals with the generalized computability and definability. Classical recursion theory focuses on the computability of functions from one natural numbers to another. A new finding in recursion theory confirms the study of application like randomness, computable model and reverse mathematics and results in pure recursion theory.
4. Proof theory and constructive mathematics is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. More so, constructive mathematics is the study of system in non-classical logic such as intuitionistic logic as well as the study of predictive systems.

The emergence of mathematical logic was the combination of formal philosophical reasoning and mathematics, (Ferreiros 2001, 443).[21] The various names for mathematical logic include,

1. Logistic
2. Symbolic logic
3. Algebra of logic
4. Formal logic

The mathematical field of category theory uses many formal axiomatic methods and includes the study of categorical logic but category theory is not considered a subfield of mathematical logic. Algebraist like George Peacock completed the traditional Aristotelian doctrine of logic into adequate framework for the learning of basics of mathematics, (Katz 1998, 686).[22] It was through

the axiomatic system that fundamental areas of mathematics such as, Arithmetics [theory of natural numbers], Analysis and Geometry were developed.

In furtherance, Cohen's proof developed the method of forcing used set theory which is now an existential tool for establishing independent results in set theory, (Kanamori, 2008).[23] It was accepted that first order logic or reasoning which is formal logic is the dominant critical reasoning used by mathematicians. In fact, mathematical logic deals with mathematical concept expressed using critical thinking system. First order logic is a peculiar formal system of logical reasoning. The syntax involves finite expression and formed formulas and the semantics are characterized by the limitations of all quantifiers. Hence, prepositional logic and first order logic are widely studied because of its applicability to the foundations of mathematics and appreciable proof of theoretical properties, (Ferreiros 2001, 441-484).[24] In summary, mathematical logic comprises two distinct areas of research;

a. The application of the techniques of formal logic to mathematics and mathematical reasoning.
b. The application of mathematical techniques to the representation and analysis of formal logic, (Stoylar 1983, 3)[25].

Mathematical logic is an extension of symbolic logic into other areas in particular to the study of model theory, proof theory, set theory and computability theory. Critical reasoning and mathematics are connected that they cannot do without each other. It means that critical reasoning is involved in the proceedings of mathematics. Having said this, it is mostly accepted that critical reasoning or thinking is an offshoot of mathematics. Mathematics is defined as a science concerned with logical reasoning. It means that mathematical statements have the similar language as logic's unique language. The use of critical reasoning in the language of statement would help while working with mathematics. This paper therefore, appraises Rene Descartes' rationalism as a mode for developing critical thinking for learning mathematics in senior secondary school. It identified some aspects of critical thinking skills which should be developed for learning mathematics in secondary schools. Therefore, the mastery of logical reasoning or thinking is the foundation for easier learning of mathematics.

Another concept used in place of Secondary School is post-primary education. Secondary education is the formal education children received after primary education and before tertiary stage. It is the education system where individuals are trained for useful society living and higher education. It is education for all, irrespective of sex, social status, religion and ethnic group. It also provides academic, technical and vocational skills for development of all aspects of life. In order to achieve the stated goals, secondary education system involves six (6) years duration given in two stages in Nigeria. They are junior and senior secondary school stages of three (3) years each. Secondary education has a diversified curriculum of studies for core and elective subjects like English language, Mathematics, Indigenous languages, Sciences and Arts. (National Policy on Education, 2014).[26]

3. Review of related literature

This section focuses on acknowledging some of the existing theoretical and empirical concepts related to the study under investigation.

Gafoor Abdul K and Kurukkan Abidha conducted a survey on 51 standard IX s students (26 boys and 25 girls) to obtain data on their mathematics related expectancies. Amongst the students, 88% selected mathematics as hated subject and 6% reported they like mathematics (Gafoor and Kurukkan, 2015).[27] The three reasons enumerated for hating the subject include:

1. Difficulty in understanding the subject matter
2. Teachers related factors
3. Instructional related factors

The ratio of 20% rated it as very difficult, 54% as medium difficulty and 10% consider it as an easy subject. This study created a gap because it focused mainly on the mathematical related expectancy. It explained how students hated mathematics and reasons for the hatred only. This seminar paper includes the development of skills of critical thinking for learning mathematics to supply for the gap created by the previous study. Hence, the two studies are related because they focused on the learning of mathematics among students.

Again this paper adopted the study by Rameli Mohd, Rustam Mohd and Kosnin Mohd Azlina to explore the types of achievement goals orientation accepted by the students for learning mathematics in Malaysian setting. A total of 976 secondary school students were selected through stratified random sampling techniques from three states in Malaysia. Achievement goal questionnaire was adapted as the research tool (Rameli & Kosnin, 2017).[28] The result presented three [3] main factors of goal orientation;

A. Mastery goals orientation
B. Performance-approach goal orientation
C. Performance-avoidance goal orientation

This study emphasized the achievement goal of orientation of students in mathematics only but was deficient in evaluating mode of achieving the goals in mathematics. Hence, this paper will complement the space created in the above study. However, their work did not reveal skills used to attain the goals in mathematics. This seminar paper is therefore intended to fill the gap of developing skills of critical thinking for learning mathematics as achievement goal in secondary schools.

Adamopoulos, L., Viskadourakis, V., Gavalas, D., Polizos, G., & Sverkos, A. (Aristodou 2020, 99-122) carried out a study on introductory course in logic offered in secondary school in Athens, Greece.[29] The course is titled 'Logic; theory and practice' which was first introduced in 2003/2004 academic session. It involved students between the ages of 17-18 years. The course was given twice a week for the whole academic year. It was divided into two groups of 20-27 students per class. Group I comprised mathematically-orientated classes and Group II involved theoretically-orientated classes usually with good background in mathematics. The course aims generally to:

1. Develop students skill in prepositional and predicate calculi
2. To encourage students to exercise these skills in application that arises in courses in practical reasoning situation.

The students in their observation explained that it helped them build bridges among disciplines and digging deeper to discern correct structures. It helps to uncover implicit forms of reasoning and their performances which improved significantly. Calculus students found the course very helpful especially when tackling a problem and they make the right connection to the reasoning.

Students also benefit intellectual development since clarity of thought and correct forms of reasoning have value outside the classroom. They observed that the same deductive logic which enhance the mathematical reasoning also aid in other fields of life. The above study highlighted on how introductory course in logic which is the science of critical reasoning helped students to connect with various discipline of learning. On the other hand, it was deficient on how critical reasoning is not applied in learning the subject mathematics in Nigeria's secondary schools. Hence, this seminar paper will supplement the gap created in the above study. The paper focused in particular on the development of skills of critical thinking in learning mathematics. It confirms that critical thinking skills are helpful in solving mathematical problems. More so, it is important to confirm that both study focused on the development of critical reasoning in learning among students.

4. Brief Biography of Rene Descartes (1596-1650)

Rene Descartes is a famous mathematician, philosopher as well as a logician. He was born in March 31, 1596 in France at a place called *La Haye en* Touraine. Joachim his father was a council member in the Parliament and Jean Brochard his mother died at his early age. Descartes joined the Jesuit College of Henri IV in La Fleche at the age of 8. This marked the beginning step of his extensive education. At 22 years, he earned Law. His popular dictum, "I think therefore I am" gave rise to his philosophical principle which is evident in his major works:

a. Discourse on the Method (1637)
b. Meditation on First Philosophy and Principles of Philosophy (Metaphysics) with its Objections and Replies (1641)
c. Principles of Philosophy (1641)
d. Passions of the soul, on the soul (1649)

Some of Rene Descartes contributions include:

1. Introduced geometry including algebra [modern algebra, Cartesian geometry]
2. Through his law of refraction Descartes developed empirical understanding of rainbow.
3. Natural account of solar system formation

There was no account of marriage but he had a daughter Francine in 1635 through a maid where Descartes was staying in France. The young girl later died of fever at age of 5. In February 11, 1650 Rene Descartes died at Stockholm in Sweden at 53 years.

4.1 Rene Descartes Philosophies of Rationalism and Mathematics

Rene Descartes was prominent in his contribution to western philosophy and progress in mathematics. He is known as the father of analytical geometry because he had contributed majorly in geometry and algebra. In furtherance, there are major contributions of Rene Descartes in Rationalism and mathematics:

4.1.1 Rene Descartes Principles of Rationalism

Descartes laid the foundation for rationalism of the philosophical views that regard reason as the chief source and test of knowledge. He developed a philosophical system known as Cartesian philosophy which is based on skepticism. He asserted that all reliable knowledge must be built up by the use of reason through logical analysis. Descartes in his words stated that, 'what pleased most about this method was that by means of it, I was assured of using my reason in everything if not perfectly…'[30]

Rene Descartes' statement, 'Cogito ergo sum' 'I think therefore I am' became fundamental to learning. It survived the test of his methodic doubt. The treat on methodic doubt was regarded as the best known philosophical statement. The fact of being conscious is the beginning of knowledge in time and logically. Descartes believes that the above proposition is the true basis of all knowledge and all philosophy. In fact, it is the basis of ultimate fact. He insisted that this is the process of knowledge or analytic reflection carried to its utmost limit. Furthermore, Descartes confirmed that self-consciousness is first truth derived from the statement. Another truth is the self- determination to attain indubitable knowledge. In addition, intuition and innate ideas are truths that are implied in the same statement. In his explanation, he stated that if he doubted, it means someone or something is doing the doubting, therefore, that fact that he doubted has given evidence of his existence. In other words, the act of doubting one's own existence provided the basic proof of reality of one's mind. The fundamental fact is that there must be a thinking being or

entity. So, for there to be thought, the self as a thinking entity must be involved. There are four (4) possible meanings of the phrase, 'Cogito ergo sum'- I think therefore I am:[31]

1. My being or existence is the effect or product of my being conscious. My being conscious creates or produces my being. Here my consciousness is first in order of existence
2. My being conscious implies that I am and was before and in order to be conscious
3. My being conscious is the means of my knowing what my existence is, or what it means. Here my consciousness is identical with my existence. My consciousness and my being are convertible phrases.
4. My being conscious informs me that I exist or through my being conscious I know for the first time that I exist. Here my being conscious is first in order of knowledge.

Rene Descartes distinguished himself as a rationalist philosopher in some of his works already mentioned in the biography section above. In the Discourse on the method, Descartes outlined a new system of Mathematics and science. The rules for the direction of the mind affirmed his views on the method of thought. By using a set of rational principles Descartes established a system of doubt to attain certainty of knowledge. He gave the twelve-vertebrae of rational thinking which consist of three sections: a. simple propositions, b. method for dealing perfectly with problems (incomplete), c. imperfectly understood problems.

4.1.2 The Rene Descartes' twelve-vertebra backbone[32] read as follows:

1. The aim of our studies must be the direction of our mind so that it may form solid and true judgments on whatever matters that arise.

2. We must occupy ourselves only with those objects that our intellectual powers appear competent to know certainly and undoubtedly.

3. As regards any subject we propose to investigate, we must inquire not what other people have thought, or what we ourselves conjecture, but what we can clearly and manifestly perceive by intuition or deduce with certainty. Perhaps, it seems there is no other way of acquiring knowledge.

4. There is need of a method for finding out the truth.

5. Method consists entirely in the order and disposition of the objects towards which our mental vision must be directed if we would find out any truth. We shall comply with it exactly if we reduce involved and obscure propositions step by step to those that are simpler, and then starting with the intuitive apprehension of all those that are absolutely simple, attempt to ascend to the knowledge of all others by precisely similar steps.

6. In order to separate out what is quite simple from what is complex, and to arrange these matters methodically, we ought, in the case of every series in which we have deduced certain facts the one from the other, to notice which fact is simple, and to mark the interval, greater, less, or equal, which separates all the others from this.

7. If we wish our science to be complete, those matters which promote the end we have in view must one and all be scrutinized by a movement of thought which is continuous and nowhere interrupted; they must also be included in an enumeration which is both adequate and methodical.

8. If in the matters to be examined we come to a step in the series of which our understanding is not sufficiently well able to have an intuitive cognition, we must stop short there. We must make no attempt to examine what follows; thus we shall spare ourselves superfluous labour.

9. We ought to give the whole of our attention to the most insignificant and most easily mastered facts, and remain a long time in contemplation of them until we are accustomed to behold the truth clearly and distinctly.

10. In order that it may acquire sagacity, the mind should be exercised in pursuing just those inquiries of which the solution has already been found by others; and it ought to traverse in a systematic way even the most trifling of men's inventions though those ought to be preferred in which order is explained or implied.

11. If, after we have recognized intuitively a number of simple truths, we wish to draw any inference from them, it is useful to run them over in a continuous and uninterrupted act of thought, to reflect upon their relations to one another, and to grasp together distinctly a number of these propositions so far as it is possible at the same time. For this is a way of making our knowledge much more certain, and of greatly increasing the power of the mind.

12. Finally, we ought to employ all the aids of understanding, imagination, sense and memory, first for the purpose of having a distinct intuition of simple propositions; partly also in order to compare the propositions.

Later, Descartes in his work on *Discourse on the method* gave the rules used to solve profound problems. He applied this rule in resolving the problem of prove of personal existence especially in his famous statement, 'cogito ergo sum'- 'I think therefore I exist'. As a matter of fact, he was drawn to the notion of method of investigation or scientific method which aid in attaining certainty of knowledge. The scientific methods begin with doubting to clear away our old disordered schemes of knowledge so as to establish knowledge afresh on solid foundations. Descartes method consists of harnessing the abilities of the mind with a particular set of rules. It is these rules by which our capacities of intuition and dedication are adequately guided in an orderly manner. The method deployed by Descartes is the kind of systematic skepticism aimed at withholding belief from anything of which we do not have a clear and distinct idea. It was far from the previous skepticism because it is assumed that something would turn out to be indubitable and the knowledge would be safeguarded. Descartes also followed these rules he outlined with persistent resolve to attain rational thinking. Again, he asserted that in order to build methodically, coherent system, we should follow four (4) specific rules of inquiry they are;

1. **First Rule**: Accept nothing as true which you did not clearly recognized to be so.
2. **Second Rule**: Divide up each difficulty into many parts in order that it might be resolved in the best possible manner.
3. **Third Rule**: Carry on the reflection in due order from the most simple to understand to the most complex
4. **Fourth Rule**: it is important to make enumeration so to complete that nothing should be omitted.[33]

The main focus of Descartes rule is a clear procedure for the operation of the mind. The rule consists in ordering and disposing of those objects in which the attention of the mind must be focused, that is, if we are to discover the indubitable (truth). The major purpose of the rule is to investigate a subject. Descartes clearly said that;

> Our inquiries should be directed not to what others have thought nor to what we ourselves conjecture, but to what we can clearly and perspicuously behold and with certainty deduce.[34]

According to Descartes, if these rules are properly followed and kept, one will never take anything false to be true. In other words, those who apply these rules will attain true understanding of all those things which do not surpass our capacity. Copleston (2003) added that Descartes does not mean to imply that there is a technique which can be applied in such a way that the natural capacities of human mind are irrelevant.[35] It means that the meaning of these rules is not immediately evident. Descartes insists that truth can be attained and we can be assured of reliable foundation for our deduction. He therefore, set out to show that we can have certainty of knowledge in his attempt to lay an irrefutable foundation for knowledge. He searched not only about mathematical concept but about the nature of reality. It is important to end this section with what Rene Descartes said in his own words about this method. He said;

> What pleased me most about this method was that it enabled me to reason in all things, if not perfectly, at least as well as was in my power. In addition, I felt that in practicing it, my mind was gradually becoming accustomed to conceive its objects more clearly and distinctly, and since I had not directed this method to any particular subject matter, I was in hopes of applying it just as usefully to the difficulties of other sciences as I had already to those of algebra, (Mathematics).[36]

4.1.3 Methodic Doubt (meditations):

Descartes began the first meditation with the confirmation for the need to discard everything completely and start over again from the foundation. He says, 'I must once for all, by a deliberate effort rid myself of all those opinions to which I have hitherto given credence, starting entirely were far better never to think of investigating truth at all than to do so without a method'.[37] He categorically stated that skeptical doubt is the right way to begin the search for truth. In clear understanding, Descartes search for method culminated in his methodic doubt or Cartesian doubt, (Hamlyn, 1970).[38] Succinctly put, the most resolute attempt to find an indubitable starting point for our knowledge was made by Rene Descartes. He turned the table on the skeptics who doubted the possibility of knowledge by turning himself or a part of himself into a skeptic advocate. His intention was clear and he described it in this words, 'I shall at least apply myself earnestly and

freely to the general overthrow of all my former opinions…I shall be justified in setting all of them aside, if each case I find any ground whatsoever for regarding them as dubitable'.[39] Descartes through his methodic doubt proved how uncertain our knowledge is even of what seems most obvious to us. Of course, he does not mean that this tool will be in use for everyday life. He presented it as intellectual critique to winnow out his belief and determine the more certain facts. In other words, this was a way of providing indubitable propositions and validating his method of inquiry. Descartes did not doubt for the sake of doubting unlike the Skeptics but for the sake of affirming certainty. Copleston (2003) confirmed the method in these words;

> It is methodic in the sense that it is practiced not for the sake of doubting but as a preliminary stage in attainment of certainty and in sifting the true from the false, the certain from the probable, the indubitable from the doubtful.[40]

Hence, Descartes through the method of re-thinking philosophy from the start, he attempted to reexamine all his previous opinions systematically in search of finding certain and secure foundation on which to build knowledge. Therefore, he used skeptics doubt to test the firmness of the foundation of knowledge.

Although Descartes quest was clear but Karl Popper (1963) disagreed with his claims on methodic doubt. He stated that in formulating epistemological results with regard to knowledge every bit of our traditional knowledge is open to critical examination and may be discarded. However, he said that knowledge cannot start from nothing but mainly modification of earlier knowledge.[41] So, errors can be detected and eliminated by criticizing the theories or guesses of others or our own. Therefore, he suggested critical rationalism as against Descartes rationalism or intellectualism.

More then, Descartes had confirmed that the way to prove absolute certainty of the foundation of knowledge is by methodic doubt. He claims that it is the ideal way to reconstruct the edifice of knowledge. He rejected belief from mere senses and reason. Descartes affirmed that methodic doubt assures indubitable truth. Furthermore, indubitable truth for Descartes means that what you believe must be true otherwise they are mere belief and not knowledge. It means that the elements of truth and knowledge must be integrated. Hamlyn (1970) supported Descartes claims when he said that knowledge involves the truth of what is said to be known; hence, a complete account of

knowledge must involve an account of truth.[42] So, truth is accepted as a condition for knowledge when truth has evidence.

In fact, the *Discourse on the Method* remains one of the world's most influential works in philosophy together with the *First Meditations on First philosophy*. In his Principles, Descartes understood philosophy to be the 'study of wisdom' or 'the perfect knowledge of all one can know'.[43] Therefore, the theories in *Discourse and Meditations* contain his principle of rationalism from where inspiration and insight are drawn to develop critical thinking skills for learning Mathematics especially in secondary schools.

4.2 Rene Descartes' views on Mathematics

Descartes invented the influential Cartesian Co-ordinate system. In 1637, one of the groundbreaking works of Rene Descartes in Mathematics is '*La Geometrie*' that is 'Geometry'. In the theory, he stated that each point in two dimensions can be explained by only two numbers on the plane. It specifies each point uniquely in a plane by a pair of numerical coordinate which are the signed distance from the point to two fixed perpendicular directed lines measured in the same line of length. Each reference line is called a Coordinate axis or just axis of the system. The point where they meet is the origin. One of the numbers is giving the points on vertical location and the other is giving horizontal location. Thus, he invented the Cartesian coordinate method which has turned to be the foundation of Analytic geometry. And this has provided geometric interpretation for other aspects of Mathematics like:

1. Linear algebra.
2. Complex analysis.
3. Differential geometry.
4. Multivariate calculus.
5. Group theory.

Descartes introduced what became the standard algebraic notion with geometry. It stated that lower cases a, b and c are used for known quantities while x, y and z are used for unknown quantities.

More so, Descartes is popularly known as the father of analytic geometry (Sorell, 2000 P.19).[44] The Cartesian coordinate laid the foundation for geometry. It allowed the conversion of geometry into algebraic equation and vice versa. Analytic geometry is widely used in physics and engineering and also in aviation, rocketry, space science and space light. It is the foundation of the various modern fields of geometry such as in:

i. Algebraic
ii. Differential
iii. Discrete
iv. Computational geometry

Therefore, analytic geometry is considered the most important contribution of Descartes in Mathematics.

Development of calculus was another area of Mathematics expanded by Descartes. His mathematical work laid the basis for the development of calculus by Isaac Newton and Gottfried Leibniz. In addition to analytic geometry, Descartes developed his rule of signs. This is the technique for determining the number of positive or negative real root of a polynomial. He was the first to use a standard notation for the superscript to denote powers. It means he was the first to denote the variable x squared as X^2. In fact, he stated that amicable numbers are two different numbers so related that the sum of the proper divisors of each is equal to the other number. Descartes independently discovered the general formula for amicable numbers known as Thabit Ibn Quarra's formula,[45] the Arab mathematician who discovered it several centuries before him. The philosophy of Rene Descartes has helped to develop the science of Mathematics. Therefore, some of his contributions in Mathematics include; Cartesian coordinate system, Fibred category, Cartesian product, Defect (geometry), Descartes rule of signs, Descartes theorem, Analytic geometry and Pullback theorem. So, it is important to conclude this section with Descartes affectionate expression for mathematics, 'I delight most of all in Mathematics because of the certainty and the evidence of its reasoning'.[46]

4.3 Relationship between Critical thinking and Mathematics

Descartes looks at mathematics as the best example of clear and precise thinking. The fact is that if the methodic doubt is employed in geometry, the method for deriving theorems by strict deductive means from intuitively obvious axioms could be successful in geometry; it is also used to attain knowledge of truth. Hence, Stumpf (1994:P.26) stated that;

> In mathematics, Descartes discovered something fundamental about mental operations. Specifically, he fastened upon the minds ability to apprehend directly and clearly certain basic truth.[47]

The mind in the view of Descartes is capable of knowing some ideas with absolute clarity and distinctness. In fact, mathematical reasoning shows how we progress in an orderly way from what we do know to what we don't know. He further confirmed that the rules of method contain the primary rudiment of reason from which he could elicit the truth in every field whatever. Copleston (2003) supported this view in these words, 'the problem facing Descartes therefore, was to reconcile his actual procedure with his ideal picture of a universal science and of a universal quasi mathematical method'.[48] In other words, he meant that Descartes did not see clearly the difference between his ideal of accumulating all sciences to Mathematics and in his perspective maintains that all the various sciences are merely different ways in which same skills of reasoning and method are used. Simply put, the method of critical reasoning of truth is the same for doing Mathematics. It means that the orderly use of intuition and deduction is applicable in any science such as Mathematics. So, to attain knowledge, both in Mathematics and other sciences, the foundation is the intuition and deduction. Beardsley (2003) said that,

> Many things are known with certainty though not themselves evident, but only as they are deduced from true and known primary data by a continuous and uninterrupted movement of thought in the perspicuous intuiting of the several items.[49]

The rules of method are built upon deduction and intuition which are the certain routes to knowledge. This is because they are two sides of the same coin involved in seeking the truth. In the intuition we grasp a simple truth completely and immediately while in deduction we arrive at a truth by a process.

In summary, Descartes as a distinguished rationalist and mathematician has revealed that mathematical truth is attained through reasoning devoid of senses. He also affirmed that the Methodic doubt which is a systematic skepticism remains the source to reliable knowledge by the use of reason through logic analysis. Hence, his statement, 'I think therefore, I exist' is an analytic reflection that gives evidence to self consciousness of a thinking being. It is also a background for self determination to achieve indubitable knowledge. Descartes disclosed that a thinking being has innate and intuitive ideas that give credence to other truth such as mathematical truth. In his works Descartes outlined a guiding set of rules used to solve profound problems or attaining knowledge. He stated that the means of scientific investigative method which begins with doubting procedures is for attaining certainty of knowledge. So, the systematic skepticism with the set of rules is a clear procedure for the operation of the mind and eventually arrival to rational thinking. It means that this process of methodic doubt will help to develop critical thinking skills especially in students which will be applied for easier learning of Mathematics.

In fact, as a father of analytic geometry who contributed immensely to geometry and algebra, Descartes presented Mathematics as an important discipline. The insight from the skeptic doubt or Methodic doubt will help to find the indubitable truth in mathematical processes. In other words, if the methodic doubt aids in attainment of certainty of truth, then, it will help in attainment of certainty of mathematical truth. The Rene Descartes Methodic doubt becomes the mode for developing the mind to be critical in the elements of truth and knowledge. Descartes rationalism presents the methodic doubt as foundation to critical thinking while his mathematical work laid the basis for the development of Mathematics. Descartes rationalism and his contribution to Mathematics confirm the relationship between Critical thinking and Mathematics. Therefore, this study had drawn fundamental insight from Descartes Principle of rationalism to develop critical thinking skills to enhance learning of Mathematics in secondary schools.

5. Descartes' Rationalism as an Insight for Developing Critical Thinking Skills:

This section deals with the development of critical thinking skills through methodic doubt which eventually will enhance the easier learning of Mathematics among students in secondary school. Descartes having established his scientific method of investigation, he proceeded to assert a system

of knowledge that can be depended on with absolute certainty. The researcher considers the process of Systematic skepticism as a means of developing critical thinking skills. The first technique is to use the systematic doubting procedure to clear away any previous belief held that was to some extent questionable. Descartes said;

> It is now some years since I detected now many were the false beliefs that I had from my earliest youth admitted as true, and how doubtful was everything I had since constructed on this basis. And from that time I was convinced that I must once for all seriously undertake to rid myself of all the opinions which I had formerly accepted and commenced to build anew from the foundation, if I wanted to establish any firm and permanent structure in the science, (Meditation 1).[50]

In this way Descartes establish the first step towards developing the thinking skill for attaining certainty of knowledge. The doubting process presents evidence of reasoning focused on discovering the truth. Of course this reasoning procedure will enhance mathematical processes.

Descartes further defined a systematic doubt as means of seeking for reason to doubt. It means that if anything can be doubted, it should be doubted. Most of the things perceived through the senses should be doubted. Descartes gave three reasons why we should doubt the senses[51]:

1. We regularly experience illusions through the senses such as when a thing is in far distance, it appears much smaller than they really are. So, it is a problem.
2. The reliability of senses is undermined when I consider the possibility of whether I am dreaming or not. For example Descartes said, a ball before me though the sense shows it exists but if I am dreaming then the experience is unreliably complete.
3. The third reason is what Descartes called Fame: for instance what if God or some evil genius is deceiving me about things even the world. He used this example to show that everything that goes on in the mind is result of a divinely implanted hallucination.

Descartes in the principle of rationalism used skeptical doubting device as a way of attaining certainty of knowledge. Hence, he stated steps to building his system of knowledge[52]:

Step One: this involves deducing details about exactly what kind of thing he is. Descartes also was able to prove his existence because he is exercising several mental abilities which he described in these words:

> But what am I? A thing which thinks, what is a thing which thinks? It is a thing which doubts, understands, conceives, affirms, denies, wills, refuses, imagines and feels… am I not that being who now doubts nearly everything, who nevertheless understands certain things, who affirms that one only is true, who denies all the others, who desires to know more, is averse from being deceived, who imagines many things, sometimes indeed despite his will and who perceives many likewise as by the intervention of the bodily organs? (Mediations 2).[53]

Descartes concludes that he is a thinking being with mental acts of doubting, understanding, feeling, refusing, willing, imagining, conceiving, denying and affirming.

Step Two: this involves prove of God's existence. The rationalist theories even from medieval philosopher offer rich argument for reality of God's existence. He began the proof by searching his mind to find collection of ideas such as trees, animals, stones, sand, etc. they are finite and unreliable. In addition, Descartes sees in his mind a concept of infinite perfection that is an idea of infinite complexity and goodness. This idea was not created by himself because his limited mental abilities would be incapable of inventing an idea that is so infinitely elaborated. The only prove is that the idea of the infinite was implanted in his mind by God himself who is infinitely perfect. Therefore, God then must exist.

Step Three: this is the step where Descartes debunked evil genius hypothesis. Descartes stated that God could not be a deceiver since deception is an imperfection. Hence, God is infinite perfection himself.

Step Four: Descartes confirms that he can have confidence in a special truth-detecting mental ability that God has endowed him. Descartes insisted that whenever he encounters obvious truth like **2+2=4**, something in his head confirms the clarity and distinctiveness. In addition to prove that he is a thinking being that exist, Descartes asserted the principle of spirit-body dualism. This is a prove that human beings are constitution of both a physical body and a spirit-mind.

In conclusion, rationalism is the philosophical view that knowledge is acquired through reason, without the aid of the senses. This rational thinking is the ability to consider the relevant variables of a situation and to access, organize, and analyze relevant information such as data, facts, opinions and judgment.[54] The same procedure is involved in critical thinking which is a gradual progression from the superficial to the increasingly complex.[55] In fact, mathematical knowledge is the best

example of rationalism. This is because through critical thinking, we can explore the depths of numerical relations, abstract proofs and deduce more complex mathematical concepts.

5.1 How to Develop Skills of Critical Thinking

It is important to begin this section with the understanding that generally Mathematics is based on structure of pure mathematics and mathematical thinking. Of course every type of statement is found in any topic in mathematics (Watson and Mason, 1998).[56] The statement connected to a particular topic in mathematics is termed its structure. Most times teachers are encouraged to help the students to focus so as to achieve higher order of mathematical thinking which is found in logical reasoning. There are three suggestions on how to stimulate a particular thinking process:

1. Take a topic and use some specific questions within certain statements and certain groups of mathematical thinking.
2. Take a mathematical process such as explaining and try to find similar examples in different topics to help you make links between topics.
3. Take a certain mathematical statement instructions and look for similar questions in different topics and help to make connection but also to understand how that statement differs within topics. In summary logical reasoning is needed in every mathematical problem.

More so, there are three basic principles that will enhance learning about thinking processes. They include:

a. Encourage students to think: The teachers have the temptation of ignoring this principle when engaging children in mathematical activities. It enhances the ability to remember. Hence, mathematical rules involve learning tricks that require only memory which is connected to one of the three of Benjamin Bloom's taxonomy of educational objective which is affective domain.[57] The children who developed this rules of memory are likely to do well while those who were not motivated to memorize lose confidence in the ability to do Mathematics. Eventually, they end up disliking mathematics. In fact, meaningful mental engagement will help the children to develop serious interest in Mathematics.

b. Encourage students to think about thinking: This can also be called 'Metacognition'. This involves thinking of how they got answer to a mathematical problem. The teacher should challenge children to think about their thinking. So as to be skilled in doing it. In the strategies used here, there are word problem. For example, consider the problem; Five (5) children want to share 25 balls of orange equally. How many oranges would each child get? While this is a proper division problem for primary 3 pupils, most nursery pupils can solve the example above without division or help from anyone (Carpenter, Ansell, et al, 1993, P. 428-441).[58] The important fact is that they were able to think about the problem and found a solution that made sense to them. They would be able to defend their logic.
c. Encourage representation of thinking: Each time the teachers encourage students to reflect on thinking and use representation of thinking in their classroom, it unavoidably leads to more thinking and more thinking about thinking. The activities expose the students to a lively and engaging classroom environment which will encourage complex and sophisticated mental activity. Therefore, it takes a lot of thinking and more thinking about thinking for both teachers and students to grow and develop especially towards learning Mathematics. Teachers themselves must learn to be thinkers before they prepare and encourage students to think. It is more important to demonstrate the logic of Mathematics than to memorize rules (Wakefield, 1998).[59]

In the field of learning, we still struggle with the same problem of math-phobia or inadequate math-teaching by teachers who were themselves taught from a surface procedure first approach. In places where success in Mathematics education is recorded, probably, it was achieved through emphasis in problem solving and conceptual approaches. It is a process where students are encouraged to think through thinking themselves and to battle with problems rather than being helped with answers by the teachers when they encounter difficulty.

6. THE STEP BY STEP PROCESSES OF CRITICAL THINKING

Descartes background in geometry and mathematics influenced his ideas on the truth and reasoning, causing him to develop a system of general reasoning now used for most mathematical reasoning. Similarly, Descartes understands that ideas could be self-evident and that reasoning alone must prove that observations are reliable. These ideas also lay strong foundations for the ideas of rationalism. In fact, there are six (6) recognized steps of critical reasoning[60] and they are:

1. **Interpretation:** This step ensures that the team or individual understands the problem by answering the questions, what, why, who, when and where, (five Ws'). These questions are asked to clarify the problem. This is to identify the current problem without emotional connection. Such questions like: a. what is happening? b. why it happened? c. who is involved? d. when it did happen? e. where did it happen?
2. **Analysis:** In this step, the individual or group discusses the problem and exploring each person's judgments, arguments, opinions and conclusions. The evidence is provided to back up the general experiences, belief, assumptions and opinion shaping the ideas. So, identifying the symptoms of the problem will help one to stay on course with the true problem. More so, it involves identify the people or processes that are fueling the problem. It deals with analysis of the facts and metrics available to corroborate the evidence. Questions like these are raised; a. what is the reason for a claim? b. what is the conclusion? c. what do you think? d. what is the basis for saying what? and e. what are your arguments?
3. **Explanation:** the individual or group formally verbalize and outline their discussion of their proposed decision and proffer solution. This step in critical thinking identifies the assumption and facts, the two different realities. The question like, what were the specific findings or results of the investigation?[61] Again explain how you conducted the analysis? Give detailed reason for a particular decision through reasoning? Therefore, the individual or group will offer discuss on the evidence for the right answer or solution.
4. **Evaluation:** this is another step used to assess the total issue involved. The individual recap the entire problem and assess reasoning and conclusions. The aim is to confirm the validity of their argument or solution, (Bent and Stubbings, 2011).[62] This is to find out whether the evidence is reliable and to what degree. It is concerned with a probe for weakness in

thinking and logic. These are some questions: a. how credible is the claim? b. how strong are those argument? c. do we have our facts right? and d. what are the consequences of this solution?

5. **Self recognition or Self-Regulation:** The main point raised here is whether any critical information is missing. More then, individual or group assesses the entire question raised so as to confirm, validate and connect the proposed decision or solution to ensure a complete process and conclusion. It involves inferential judgments of the people who form part of critical thinking. Here are some questions in this step: a. show how good is the method and the way it was followed? b. how good is the evidence? c. what are we missing before finalizing? d. finding how to revisit what we mean by certain things to clear confusion and making the final decision.
6. **Inference:** This seems like the last step. The individual or group brainstorm possible solutions using all data and information available. Other possibility that may be causing problem is identified. These important questions are asked: a. what facts are available to draw reasonable conclusions? b. what are the possible alternatives? c. what are the consequences of these conclusions? d. what additional information is explored to resolve the problem?

These steps provide a foundation for students to practice critical thinking skills.[63] This forms a reflective practice that produces deeper thinking and a meaningful conversation. It is expected that these processes will assist to resolve the difficulty in learning mathematics.

6.1 Critical thinking skills as a mode of learning Mathematics

Critical thinking holds important key as a mode of teaching mathematics in our secondary schools. The reason being that 'teaching critical reasoning in order to foster competence with proof in mathematics classroom…offer suggestions about how to modify curriculum to help students develop their logical reasoning abilities', (Viviane Durand-Guerrier, 2012).[64] The examples of critical thinking skills include comparing, contrasting, categorizing, analyzing and evaluating (ACARA, 2012).[65] These higher-order thinking skills are integral to students working mathematically, as students interpret and justify their decisions based on logical thought and

actions (BOS NSW, 2012).[66] Mathematicians use critical reasoning skills to solve problems and construct mathematical proofs, (Aufmann et al., 2007).[67] In fact, critical thinking has proved to be useful tools in mathematical processes. Generally, logical thinking is foundational to any field that makes use of arguments. It has especially close connections to mathematics, computer science etc. Aristotelian logical reasoning and modern symbolic logic are impressive bodies of knowledge that constitute major intellectual achievement (Westacott, 2019).[68] In general terms, logic refers to the philosophical study of correct reasoning. It deals with principles of sound argument. And no doubt, in various forms of learning especially in mathematics, there are forms of argument where statements are made and conclusions drawn. The philosophy of critical reasoning of Rene Descartes has proffered modes for learning mathematics which can be applied on secondary school students. This is because the students find the learning of mathematics quite difficult. And Mathematics is of great interest right from the early modern mathematicians such as Descartes, who delights in mathematics because of the certainty and self-evidence of its reasoning, (Bos, 2001).[69] Below are the procedures;

6.2. Thinking in mathematical processes:

Consequently, it is necessary to re-emphasize and repeat this fact that Rene Descartes offered the first mode of learning mathematics in his famous dictum and best known philosophical statement, 'cogito ergo sum' which means 'I think, therefore I am'. He tried to establish a system of knowledge relying upon rationalization and logical deduction. In his explanation he stated that if he doubted, it means someone or something is doing the doubting therefore, the fact that he doubted has given evidence of his existence. It confirms that the act of doubting one's own existence provided the basic proof of reality of one's own mind. The fundamental fact is that there must be a thinking being or entity. So, for there to be thought, the self as a thinking entity must be involved. Hence, he understood reasoning as the only way to attain higher knowledge. He laid the foundation for rationalism in his philosophical views that regard reason as the chief source and test of knowledge. By this fact, Descartes suggests that skills of reasoning or logical thinking should be taught as a subject of course to the students in countries where mathematics education is considered fundamental. One stress is always on problem-solving and conceptual approaches. This is because students are encouraged to think through and struggle with the problem rather than finding other easier shortcut to the solution. In essence, the students should be taught reasoning

skills which will help them to form logical arguments within themselves. Hence, thinking logically is a foundation stone to learning mathematics. In fact, lessons on reasoning skill will introduce several types of reasoning such as:

1. Connective reasoning.
2. Inductive reasoning.
3. Deductive reasoning.
4. Abstract reasoning.
5. Quantitative reasoning.
6. Adaptive reasoning.

These forms of reasoning will dispose the students for learning the mathematical procedures. It will introduce the use of direct proof method which involves definition and application. Teachers who teach mathematics should include problem-based learning technique which is a method of making learning engaging and important. The teacher should use sense making which is using basic logic in teaching mathematics. It involves two things:

a. Number sense: It is the intuitive understanding of magnitude, range and estimates of numerical value based on both reasoning and experience. It is important and critical in processes involving deep reasoning. It helps the students to identify incorrect possibilities before proceeding in the processes of the problem-solving.
b. Estimation is identifying the approximate correct values to a mathematical problem and a very important step in sense making. A good example is a child knowing how many hands or feet each person has and how many fingers are on each hand. This is a great basic exercise to develop number sense.

There is no doubt whether critical reasoning is needed in many mathematical problems. Deductive reasoning will enhance learning of mathematics among students. This is because mathematics is mainly made of deductive reasoning (Epp, 2003).[70] Descartes introduced the rules of deductive logic which can aid a student in learning mathematics:

1. Accept nothing as true that is not self-evident.
2. Divide problems into their simplest parts.
3. Solve problems by proceeding from simple to complex.

4. Recheck the reasoning, (Britannica, 2019).[71]

These rules are direct application of mathematical procedures. Therefore, he asserted that all reliable knowledge must be built up by the use of reason through logical analysis. Critical thinking and reasoning allows students to think about how they utilize their mathematical skills. Met-cognition helps students in recognition of solutions to problems, (Huang, et al., 2016).[72]

6.3 Language of Critical thinking is language of Mathematics:

Critical thinking is a mode for teaching mathematics because it has its own peculiar language and means of defining what is true and false. The language of critical reasoning can help a child to critically think while working with mathematical problems. As previously mentioned, Descartes was the first to assign alphabets like a, b and c to represent data and numerical quantities while alphabets like x, y and z are used to represent unknown quantities and variables. Again, Descartes was the first to use a standard notation for the superscript to denote powers such as variable X squared as X^2. He perfected the use of x, y and z for squares, cubes and other powers with numerical superscript as $X^2$and X^3. This made algebraic calculation easier than they had been before, (Watson, 1998).[73]

6.4 Critical thinking as tool for mathematical problems:

Critical thinking can enhance creative problem solving options by encouraging students to seek new strategies when solving mathematical problems. Critical reasoning helps to build the skills required for higher level mathematics. It was stated that improving critical thinking abilities require early and constant practice and to be actively engaged in skill of thinking critically (Gelder, 2005).[74] More so, the suggestion for improving critical thinking includes: practice of active engagement, transfer of learning, understanding theories, thinking map skills etc (Gelder, 2001).[75] Some scholars affirm in their studies that students can expand their thinking skills through their precision, correctness, accuracy, importance, intensity, extent and reason, (Reichenbach, 2001).[76]

6.5 Critical thinking and Mathematical methods:

Mathematical proceedings require methodical processes. The method is used in finding the truth of mathematical set up. The methods consist of the order and disposition of the subject and object towards which the mental vision must be directed in order to find the truth. The mind focuses its direction on the proceeding to form solid and true judgment in solving the mathematical problem. In exercising critical thinking, the intellectual power of the mind is applied to the arithmetical problem to separate the simple from the complex methodically. In doing this, the process of scrutiny is ongoing and movement of thought remains continuous and uninterrupted. It means that deep attention must be intended so as to master the facts and be accustomed with the process to arrive at clear and distinct truth. The mind should be continuously engaged in inquiries to discover the solution to the problem which is the aim. In this whole process, the intuitive cognition engages in recognition and reflection of all the simple facts and truth. This will help to know the relation of every step with one another and as such certainty of knowledge is assured. More so, one should employ the aids of understanding, imagination, sense and memory so as to have distinct intuition of simple proposition involved in mathematical structure. In this way, critical thinking skills become the tool used by students for learning mathematics in the secondary schools.

7. A Critique of Critical Thinking as a Mode of Learning Mathematics:

There is no doubt that students find the learning of mathematics quite difficult. It has been an age old puzzle to solve. Perhaps, this paper has tried to proffer the Cartesian Critical thinking of Rene Descartes as the novel means of developing skills for learning mathematics. The important question now is, can the critical thinking skill solve the problem encountered in learning mathematics for secondary school students? I think learning the skill of critical thinking form a fundamental basis for learning mathematical processes. It involves critical reasoning and so, it will be benefittial in solving mathematical problems. Descartes suggested that to engage in acquiring knowledge, the power of the mind which is critical reasoning is applied as implied in his dictum, ‘Corgito ego sum’-I think, therefore, I exist’. However, there are challenges to acquiring and application of critical reasoning skills. The first observation is whether we have the capable human resources to teach the skills of critical thinking as subject in the schools? More so, if we proffer

the suggestion to teach critical thinking to the students from the early age, do we have qualified teachers with the ability to impart the skills of critical reasoning? Of course, it seems that there are no enough capable teachers to handle the proposed subject of critical thinking. The difficulty of formulating curriculum and scheme of work may constitute a problem. Again, the processes of teaching critical thinking to the students may appear to be complicated like mathematics or rather it seem more confusing than mathematics. It is possible that the secondary school students do not have the capacity for critical reasoning to the level of learning Cartesian critical thinking. It may ends up constituting more problems other than helping in learning mathematics. But I should think that a secondary school student has attained a reasonable level of mental maturity. Teaching them skills of critical thinking at this stage will not be a serious problem. Rather, I am foreseeing the problem of lack of students' disposition and interest in acquiring such skills. Since, they find it difficult and disinterested in learning mathematics, the same problem may arise in learning skills of critical thinking. This is because of the rigors and abstractions involved in the learning process. Therefore, it will be difficult but possible to teach and learn the critical thinking skills in secondary schools. The one suggestion here is to find a way to teach it, so, that critical thinking will form the good foundation for learning mathematics. More so, Cartesian critical thinking is presented in this paper as the best means of developing skills for learning mathematics. But it doesn't mean that, there are no other pre-existing alternative modes of teaching and learning critical reasoning, which may be interesting and appealing to the students. I know for sure that there are aptitude and qualitative subjects which helps to test the reasoning capability of students. There are puzzle games like Chess, Draft, Cross-word and puzzle toys. There are other puzzle games like 3D Tetris, Alpha Twist, Arrange puzzle, blob game, break-lock, broken calculator, Cubilus game and so on. These puzzle games are used to test problem solving skills including Logic, sequences solving etc. In order words, puzzle games focus on logical and conceptual challenges, (Rollings and Adams, 2006).[77] It is important to state that the puzzle game was developed to address the need to enhance students learning in the classroom (Elson, Ostapski, et al., 2012).[78] In fact, there are three basics of what puzzle do for children. It develops their physical, cognitive and emotional skills, (Manno, 2013).[79] Since these games help to develop logical reasoning, of what essence is Cartesian critical reasoning? Of course, these games offer their important aid but Critical thinking has proven to be an authentic skill of logical reasoning. It proved to be a fundamental skill in search for knowledge. It needs to be adopted and adapted to the various levels of the students. If it is applied at the

different levels and ages of the students, the application and resultant effect will be productive in learning mathematics. It will broaden the reasoning capacity of the students beyond what the puzzle game will offer. This is because Cartesian critical reasoning is a typical logic. It is a skill of high standard that can match the mathematical processes. Observations confirmed that many children play these games well but does not perform well in mathematics classes. In other words, the puzzle games do not give the entire basic skills of critical reasoning. In the other hand, the principles of critical reasoning learnt from the Cartesian theories will probably lay more appropriate foundation for excelling in mathematics. Therefore, it is affirmed that the abstract, creative, innovative and communicative thinking acquired from Rene Descartes principles of critical reasoning will provide the fundamental requirements for learning mathematics.

Practical application of Descartes Rationalism for learning Mathematics in schools

The application of critical thinking in mathematics is nothing other than the process of critical thinking which relate to knowledge of mathematics, mathematical reasoning and mathematical proofs in mathematical problem-solving (Krulik & Rudnick in Firdaus et al., 2015)[80]. Some researchers agree with the purpose of this paper which is the fact that development of critical thinking skill enhances mathematical achievements. The essence is for students to develop quality independent and problem-solving abilities in school especially in subjects like mathematics (Firdaus et al., 2015)[81]. The particular aspects of Descartes Rationalism that will practically help the students:

1. Quality Thought: Developing critical thinking skills in student will improve their ability for mathematical thinking. Simply put, it improves students' high thinking ability. It also develops in students the processes of interpretation, analysis, evaluation, inference, explanation and self-regulation (Facione, 2011)[82]. The quality thought developed in students will help to understand the mathematical content as well as the processes of mathematical processes (Rajendran, 2010)[83]. It means that the processes are infused in the teaching method of schools. Teachers teach students how to think. This practical application of developing quality thought is drawn from Descartes philosophy which consisted of method of systematic doubt. It means the process of rebuilding knowledge from the ground. This is exemplified in his popular statement, 'cogito ergo sum'- 'I think therefore I am'.
2. Rules of sign: the greatest contribution of Descartes to philosophy and mathematics is the rule of sign. This is the techniques for determining the number of positive or negative real roots of polynomial. Example is 2^4 to show $2\times 2 \times 2 \times 2 = 16$. The rule of sign helps to demonstrate thoughts into symbols. Hence, the signs become the visible representation of invisible thoughts which is the critical thinking process conceived through reason.

3. Deductive and inductive reasoning: critical thinking process involves both method of deduction and induction. Descartes insisted that certainty of knowledge is possible through these processes of reasoning. They form the two major forms of reasoning. In the first place, Descartes agreed that mathematical methods are coordinated with the aid of reason. Reason is capable of attaining truth or indubitable knowledge without making reference to the senses (Ihejirika & Wolemonwu, 2015)[84]. So, Descartes adopted the mathematical method of induction and deduction which means orderly, logical reasoning or inference from self-evident proposition. These processes will guarantee certainty of knowledge or truth that provides clear and distinct perception of the objects by the subject in the light of reason. However, the use of inductive and deductive reasoning is popular in solving mathematical problems. The deductive reasoning is always based on facts while the inductive reasoning relies on a group of observations. Deduction is going from general to particular while induction is going from particular to general. It is used in geometrical processes and other forms of mathematical processes.

8. Conclusion

Mathematics has been perceived by majority of students as a difficult subject that should be eliminated from the curriculum of study. However, the relevance of mathematics to national and global development cannot be quantified. It connects all the field of studies such as economics, computer analysis, architecture, medical sciences, natural sciences and so on. It forms the basis of all human living. The solution is not to remove mathematics from field of learning but to tackle the problem of why it is difficult in learning. This paper, therefore, had proffered the solution of developing the skills of critical thinking from early age to advance level of study in order to form the right minds for mathematical studies. Critical thinking skills will help to demystify the myths built around mathematics that it is difficult and make it easier for learning. In spite of all the challenges presumed to hinder the benefit of critical thinking skills, it will help students to understand and resolve the logical language and complexities perceived in mathematics. This seminar paper has appraised to an extent the rationalism of Rene Descartes as a mode for developing critical thinking skills for learning mathematics in secondary schools.

9. Endnotes

[1] Singh Kubir, *The Teachers of Mathematics*, (New Delhi: Sterling Publishers Private Ltd (1), 2006).

[2] Maduabum, M. A & Odili, G.A. "Analysis of students' performance in General mathematics at SSCE level in Nigeria between 1992 to 2002". *Journal Research for curriculum teaching* 1(11)(1), (2006), 64-68, 2006.

[3] Osarenren, N & Asiedu, A.S. "Counselling Implication on the Role of the New Mathematics Teacher in the Teaching and Learning Mathematics". *International Journal* (1), (2007).

[4] Rene Descartes, *Descartes Rene, Oeuvres Descartes*, eds. Charles Adam and Paul Tannery, (Paris: Vrin Cited in the Text as AT volume), 1913-1987

[5]Robert Hanna, *Kant and foundations of analytic philosophy*, (United Kingdom: Oxford Press, 2011), 196 https://www.academia.edu Accessed June 27, 2021

[6] Rahul Semil, "Enhancing thinking skills in the classroom". *Human and social sciences Journal* 1 (1), (2006), 28-29

[7] Mansoor, F. & Pezeshki, M. Manipulating critical thinking skills in test taking, *International Journal of education* 4 (1), (2012), 153-160

[8] Heard, J., Scoular C., Duckworth, D et al., "*Critical Thinking: Skill Development Framework", (Australian Council for Education Research*, 2020), 6 https://research.acer.edu.au/ar_misc/41, Accessed July 3, 2021.

[9] *Stanford Encyclopedia of Philosophy* (2018). https://www.plato.stanford.edu, Accessed December 10, 2020

[10] Paul R, and Elder L, The Miniature Guide to Critical Thinking Concept and Tools, (Dillon Beach: Foundation for Critical Thinking Press, 2010)
https://louisville.edu/ideastoaction/about/criticalthinking/framework Accessed July 27, 2021

[11] Fisher A & Michael Scriven, Critical Thinking: it Definition and assessment, (United Kingdom: Centre for research in Critical Thinking, Edge Press, 1997)

[12] Udigwomen, A.F. *Philosophy and Rise of Modern Science*, (Nigeria: El-John Publisher, 2011), 304-309

[13] Udigwomen, A.F. *Philosophy and Rise of Modern Science*, Nigeria, 2011.

[14] Kim Denley, & Mike Hall. *Viewing life Mathematically-A Pathway of Quantitative Literacy*, (USA: Hawkes Learning system, 2016).

[15] Copi Irving, & Carl Cohen. *Introduction to Logic*, 11th Edition, (India: Prentice Hall Private Limited, 2001).

[16] Anne Watson & John Mason. *Questions and Prompt for Mathematical Thinking*, (Derby: Association of teachers of mathematics, 1998).

[17] Jeremy, J G., Craig, G., Frasher, J.L., et alia. "Mathematics" *Encyclopedia Britannica* Inc, (2019), https://www.britannica.com Accessed January 7, 2021.

[18] Jonathan Barnes. *The Cambridge Companion to Aristotle*, (USA: Cambridge University Press, 1995), 27

[19] Alfred Tarski. *A Decision Method for Elementary Algebra and Geometry*, (California: Santa Monica, RAND Corporation, 1948).

[20] Lou Van Dries. *Mathematical Logic Lecture Notes*, (2016), P.2, https://facultymath.illinois.edu Accessed May 3, 2021

[21] Jose Ferreriros. "The road to modern logic-An interpretation", Journal article, *The Bulletin of symbolic logic* Vol.7, (Association for Symbolic logic, 2001), 443.

[22] Victor Katz. *A History of Mathematics*, 2nd Edition, (New York: Addison-Wisley Company Incorporated, 1998), 686.

[23] Aikihiro Kanamori. "Cohen and set theory" Journal Article, *The Bulletin of Symbolic Logic*, Vol. 14, No.3, (2008), P. 351-378

[24] Jose Ferreiros. "The road to modern logic-An interpretation", Journal article, 441-484

[25] Abram Stoylar. *Introduction to Elementary Mathematical Logic*, (USA: Dover Publications, 1983), 3

[26]Federal Republic of Nigeria, *National policy on education*, (Lagos: Ministry of information, Printing Division, 2014), 14-18.

[27] Abdul Gafoor and Abidha Kurukkan. "Why high school students feel mathematics difficult? An exploration of effective belief online submission", A paper presented at the UGC sponsored national seminar on *Pedagogy of Teacher Trend and Challenges*, (India: 2015) https://eric.ed.gov Accessed April 5, 2021

[28] Mohd Rustam Rameli & Azlina Kosnin. "*The survey on mathematics achievement goal orientation among Malaysia students"(Application of rasch measurement*, India: 2017), https://www.researchgate.net Accessed on June 3,2021

[29] Michael Aristidou. "Is Mathematics Logic Really Necessary in Teaching Mathematical Proofs?" *Athens Journal of Education* Vol. 7 Issue (1), (2020), 99-122 https://doi.org/10.30958/aje.7-1-5 Accessed on June 15, 2021

[30] Rene Descartes. *Discourse on method and meditations on First Philosophy* 4th edition, Translated by Donald A. (Cress Indianapolis/Cambridge: Hackett Publishing Company, 1998), 12

[31] Rene Descartes. *The Method, Meditations and Philosophy*, Translated from the original texts, with a new introductory, essay, historical and critical by John Veitch, L.L.D, (Washington: M. Walter Dunne Publisher, 1901), https://oll.libertyfund.org, Accessed 3/9/21

[32] Maria Popova. *Rules of the Mind: Descartes' 12 Timeless Tenets of Critical Thinking*, https://www.brainpickings.org Accessed September 4, 2021

[33] Ian Maclean. *Rene Desacartes A Discourse on the Method,* A New Translation, (USA: Oxford University Press, 2006), 17

[34] Samuel Enoch Stumpf. Philosophy: *History and Problems*, (New York: Mc-Graw Hill, 5th Edition, 1994), 227-228

[35] Frederick Copleston. *History of Philosophy*, Vol.4, (New York: Continuum Book, 2003), 72

[36] Maclean. *Rene Descartes: A Discourse on the Method, A New Translation*, (USA: Oxford University Press, 2006), 19.

[37] Rene Descartes. "*Rules for directing the mind" in the Philosophical works of Descartes* Vol.1(Trans.) Haldane E.D and Ross G.R.T, (Cambridge: Murray Printing Co., 1911), 9.

[38] David W Hamlyn. *The theory of knowledge,* (London: the Macmillan Press Ltd, 1970).

[39] Monroe Curtis Beardsley. *The European Philosophers from Descartes to Nietzsche*, (Mordern Library 2002), 33

[40] Copleston, *History of Philosophy*, Vol.4, New York: Continuum Book, 2003, 85

[41] Karl Popper. *Conjectures and Refutation*, (London: Routledge, 1963), 26

[42] Hamlyn. *The theory of knowledge*, London: the Macmillan Press Ltd, 1970.

[43] *Britannica*. Online publication, 2021, www.britannica.com Accessed September 1, 2021

[44] Sorell, T. *Descartes: A very short introduction*, (New York: Oxford University Press, 2000).

[45] Amicable numbers. https://mathworld.wolfran.com/ThabitibnKurrahRule.html Accessed on September 4, 2021

[46] Rene Descartes. *Discourse on method and meditations on First Philosophy* 4th edition, Translated by Donald A. (Cress Indianapolis/Cambridge: Hackett Publishing Company, 1998), 4

[47] Stumpf. *Philosophy: History and Problems*, 1994, 26

[48] Copleston. *History of Philosophy*, Vol.4, New York: Continuum Book, 2003, P.72

[49] Beardsley. *The European Philosophers from Descartes to Nietzsche*, 2002, 80-81

[50] Renati Des-Cartes. *Meditationes, De Prima Philosophia in Qua Dei Existentia et Animae Immortalitas Demonstratyr*, Parisii, Apud Michaelem Soly via Iacobea, fub figno Phoenicis M.DC. XLI cum Priuilegio Approbatione Dictorum. This file is of the 1911 edition of the Philosophical works of Descartes Translated by Elizabeth, S. Haldene, (Cambridge: Cambridge University Press, Internet Encyclopedia of Philosophy 1996),1-32.

[51] Renati Des-Cartes. *Meditationes, De Prima Philosophia in Qua Dei Existentia et Animae Immortalitas Demonstratyr*, Parisii, Apud Michaelem Soly via Iacobea, fub figno Phoenicis M.DC. XLI cum Priuilegio Approbatione Dictorum, 1996

[52] James Fieser. *Continental Rationalism from the History of Philosophy: A short survey*, (2020), https://www.utm.edu Accessed September 1, 2021.

[53] Renati Des-Cartes. *Meditationes, De Prima Philosophia in Qua Dei Existentia et Animae Immortalitas Demonstratyr*, Parisii, Apud Michaelem Soly via Iacobea, fub figno Phoenicis M.DC. XLI cum Priuilegio Approbatione Dictorum. This file is of the 1911 edition of the Philosophical works of Descartes Translated by Elizabeth, S. Haldene, Cambridge: Cambridge University Press, Internet Encyclopedia of Philosophy 1996

[54] *Analytic Processes for School Leaders*, https://www.ascd.org>books Accessed August 30, 2021

[55] Lisa Mendelman. "Critical thinking and reading". *Journal of Adolescent and Adult Literacy, 51(4)*, (2007), 300-304, http://www.reading.org/General/Publications/Journals/jaal.aspx44, accessed August 26, 2021

[56] Anne Watson, & John Mason. *Questions and Prompt for Mathematical Thinking*, (Derby: Association of teachers of mathematics, 1998).

[57] Patrick Eya & Roseline Chidiobi. *Methods in Teaching (Best Practices)*, (Enugu: God's Will Prints Enter, 2012), 114

[58]Carpenter, T.P., Ansell, E., Franke M.L., Fennema E, & Weisbeck. "Model of problem solving; A study of Kindergarten Children's Problems solving processes", *Journal for research in mathematics education* Vol. 24 No. 5, (1993), 428-441

[59] Ali Wakefield. *Early childhood number games, teacher reinvents math instruction Pre-kindergarten through 3rd Grade*, (Boston: Allyn and Bacon, 1998).

[60] Jonathan Heard, Claire Scoular, Daniel Duckworth, Dara Ramalingan & Ian Teo. "*Critical Thinking: Skill Development Framework", (Australian Council for Education Research*, 2020), 6 https://research.acer.edu.au/ar_misc/41, Accessed July 3, 2021.

[61] Robert Ennis. *Critical Thinking Across the Curriculum: (A vision*, Topoi 37, 2018), P.165-184

[62] Moira Bent & Ruth Stubbings. *The SCONUL Seven Pillars of Information Literacy: The Core Investigating Critical Thinking Skill of Junior High School Model*. In SCONUL (Ed.) (SCONUL working group on information literacy SCONUL, 2011).

[63] Keith Thomas. & Beatrice Lok. *Teaching Critical Thinking: An Operational Framework in M. Davies and R. Bernett (Eds).* The Palgrave handbook of critical thinking in higher education, (USA: Palgrave Macmillan, 2015), 93-105.

[64] Viviane, Durand-Guerrier., Paolo, B., Nadia, D., et alia., *Examining the Role of Logic in Teaching Proof*, Vol.15, (New ICMI study series, 2012).

[65] Australian Curriculum Assessmemnt and Reporting Authority, (ACARA). (*The shape of Australian Curriculum*, 2012), https://www,acara.edu.au/verve/resource/the_shape_of _the_australian_curriculum_v4.pdf Accessed 18, February, 2019.

[66] Board of Studies, New South Wales (BOS NSW), *Mathematics K-20 Syllabus*; NSW syllabus for the Australian curriculum, 2012, (Sydney: Author Yuliani & Saragih 2015), 180-200

[67] Richard Aufmann, Jaonna Lockwood, Richard Nation & Daniel Clegg. *Mathematical Excursion*, 2nd Edition, (New York: Houghton Mifflin company, 2007).

[68] Emry Westacott. *5 Good Reasons to Study Logic*, 2019, Retrieved from https://www.thoughtco.com/good-reason-to-study-logic-2670416 Accessed on December 20, 2021

[69] Bos, Henk. *Redefining Geometry Exactness, Descartes Transformation of the Early Modern Concept of Construction*, (Berlin: Heidelberg Springer-verlag, 2001), 6

[70] Sussana Epp. "The role of logic in teaching proof", *Journal of mathematics Association of America* Vol. 110, No.10, (2003).

[71] Maclean, Rene Descartes: A Discourse on the method, (New York: Oxford University Press, 2006), 17

[72] Hui Fang Huang Su; Frederick Ricci, & Mamikon Mnatsakania. Mathematical teaching strategies, pathway to critical thinking and Metacognition, *International Journal of Research in Education and Science* Vol. II No1, 2016, P.190-200

[73] Watson & Mason, Questions and prompt for mathematical thinking, (Derby: Association of teachers of mathematics, 1998).

[74] Tim Van Gelder. *Teaching Critical Thinking, Some Lessons from Cognitive Science College*, (Published Online 45, 2010), 41-48, https://doi.org/10.3200/CTCH.53.1.41-48 accessed on July 27, 2021

[75] Tim Van Gelder. "How to improve critical thinking using educational technology, meeting at the crossroads" (Proceeding of the 18th Annual Conference of the Australasian society for *Computers in Learning in Tertiary Education*, 2001), Retrieved from https://www4.uwm.edu/org/mmp/ACm201213-file/ACm-march15BloomRevisedmath.pdf accessed on May 3, 2021

[76] Bruce Reichenbach. *Introduction to Critical Thinking*, (Washington DC: American Psychological Associaton, 2001).

[77] Ernest Adams & Andrew Rollings. *Fundamentals of Game Design: Game Design and Development Series*, (USA: Prentice Hall Inc, 2006).

[78] Elson, R.J., Ostapski, S. A., Callaghan, S.O and Walker J.P. "Enhancing the understanding of government and non-profit accounting with the puzzle game", A pilot study Journal of instructional Pedagogies 9, (3), (2012), P. 1-6

[79] Michelle Manno. The benefits of puzzle in early childhood development, (2013) https://teach.com Accessed on July 11, 2021

[80] Firdaus, F., Kailani, I., Bakar, N. B & Bakry, B. Developing critical thinking skills of students in mathematics learning, (2015), www.researchgate.net, accessed on June 11, 2022

[81] Firdaus et al., Developing Critical thinking skills of students in mathematics learning

[82] Facione, P.A. Critical thinking (Millbrae, CA: The California Academic Press, 2011).

[83] Rajendran, N.S. Teaching and acquiring higher order thinking skills (Perak: Penerbit Univesiti Pendidikan Sultan Idris, 2010).

[84] Ihejirika, C & Wolemonwu, V. The role of the subject in Rene Descartes theory of Knowledge, 2015, www.researchgate.net, Accessed on June 11, 2022

9.1 Bibliography

Amicable numbers. https://mathworld.wolfran.com/ThabitibnKurrahRule.html Accessed on September 4, 2021

Analysis of students Performance in WASSCE on The Management of Public Senior Secondary Schools in Nigeria (2011-2015) https://www.projectsxtra.com Accessed on May 25, 2021

Analytic Processes for School Leaders, https://www.ascd.org>books Accessed August 30, 2021

Aristidou, Michael. "Is mathematical logic really necessary in teaching mathematical proofs?" *Athens Journal of Education* Vol 7 Issue(1), (2020), 99-122. https://doi.org/10.30958/aje.7-1-5, Accessed June 15, 2021

Australian Curriculum, Assessment and Reporting Authority (ACARA). *The Shape of the Australian Curriculum.* , 2012, URL: http://www.acara.edu.au/verve/_resources/the_shape_of_the_australian_curriculum_v4.pdf (Accessed February 18, 2019).

Aufmann, Richard, Lockwood Joannes, Nation Richard and Clegg Daniel. *Mathematical Excursion*, 2nd Edition, New York: Houghton Mifflin Company, 2007.

Barnes, Jonathan. *The Cambridge Companion to Aristotle*, USA: Cambridge University Press ISBN 978-0-521-42294-4, 1995, 27.

Beardsley Monroe Curtis. *The European Philosophers from Descartes to Nietzsche*, Modern Library, 2002, 33, 80-81

Bent, Moira and Stubbings, Ruth. *The SCONUL seven pillars of information literacy: the core investigating critical thinking skill of junior high school model*. In SCONUL (Ed.) SCONUL working group on information literacy SCONUL, 2011.

Board of Studies, New South Wales (BOS NSW). *Mathematics K–10 Syllabus: NSW syllabus for the Australian curriculum*. Sydney: Author, 2012.

Bos, Henk. (2001). *Redefining Geometry Exactness, Descartes Transformation of the Early Modern Concept of Construction*, Berlin: Heidelberg Springer-Verlag, 2001.

Britannica. *Descartes French Mathematician and Philosopher*. 2019.

Britannica. Online Publication, 2021, www.britannica.com, Accessed September 1, 2021

Carpenter, T.P., Ansell, E., Franke M.L., Fennema, E., and Weisbeck L. "Model of problem solving; A study of Kindergarten Children's Problems solving processes", *Journal for research in mathematics education* Vol. 24 No. 5, (1993), P. 428-441

Chukwuyenum, Asuai Nelson. "Impact of critical thinking on performance in mathematics among senior secondary school students in Lagos state", *Journal of Research & Method in Education* e-ISSN 2320-7388,P-ISSN 2320-737X volume 3, Issue 5 (1), (2013), 18-25 www.iosrjournals.org Accessed July 13, 2021

Copleston Frederick. *History of Philosophy*, Vol.4, New York: Continuum Book, 2003, 72, 85

Dalen, Dirk Van. *Logic and Structure*, 4th Edition, ISBN 978-3-540-20879-2, Berlin: Springer-Verlag, 2008.

Denley, Kim. and Hall, Mike. *Viewing Llife Mathematically-A Pathway to Quantitative Literacy*, ISBN 978-1-9357-82056, USA: Hawkes Learning system, 2016.

Dries, Lou Van Den. *Mathematical Logic Lecture notes*, 2016, https://faculty.math.illinois.edu, Accessed May 3, 2021

Elson, R.J., Ostapski, S. A., Callaghan, S.O and Walker J.P. "Enhancing the understanding of government and non-profit accounting with the puzzle game" *A Pilot Study Journal of Instructional Pedagogies* 9, (3), (2012), 1-6

Ennis, Robert. *Critical Thinking Across the Curriculum: A Vision*, Topoi 37, 2018, 165-184

Epp, Sussana. *The Role of Logic in Teaching Proof, Journal of Mathematics Association of America*, Volume 110 No.10, 2003.

Essay UK. [November 2018] retrieved from https://www.ukessay.com, Accessed February 20, 2021

Eya Patrick and Chidiobi Roseline. Methods in Teaching: Best Practices, Enugu: God's Will Prints Enter, 2012, 114.

Facione, P.A. Critical thinking, what it counts, Millbrae, CA: The California Academic Press, 2011.

Federal Republic of Nigeria. *National Policy on Education*, Lagos, Ministry of information, Printing Division, 2014.

Ferreiros, Jose. "The road to modern Logic-An interpretation", *Journal Article, The Bulletin of Symbolic Logic* Vol. 7 No 4, Association for Symbolic Logic, (2001).

Fieser James. *Continental Rationalism from the History of Philosophy: A short survey*, (2020), https://www.utm.edu Accessed September 1, 2021.

Firdaus, F., Kailani, I., Bakar, N. B & Bakry, B. Developing critical thinking skills of students in mathematics learning, Journal of Education and Learning, 9(3), (2015), 226, www.researchgate.net, accessed on June 11, 2022

Fisher A and Michael Scriven, Critical Thinking: it Definition and assessment, United Kingdom: Centre for research in Critical Thinking, Edge Press, 1997.

Gafoor, K. A., and Kurukkan, A. "Why High School Students Feel Mathematics Difficult? An Exploration of Affective Beliefs Online Submission", A Paper presented at the UGC sponsored national seminar on *Pedagogy of Teacher Education Trend and Challenges*, India: (2015), https://eric.ed.gov, Accessed April 5, 2021

Gelder Van Tim. Gelder Van, Tim. *Teaching Critical Thinking, Some Lessons from Cognitive Science College*, Published Online 45, 2010, 41-48, https://doi.org/10.3200/CTCH.53.1.41-48 accessed on July 27, 2021

Gelder Van Tim. "*How to Improve Critical Thinking Using Educational Technology, Meeting at the Crossroads".* Proceedings of the 18th Annual conference of the Australasian society for computers in

learning in Tertiary education (2001), Retrieved from http://www4.uwm.edu/Org/mmp/ACm201213-file/ACm-march15Bloom Revisedmath.pdf, Accessed May 3, 2021

Grondin, Jean. *Introduction to Metaphysics from Parmenides to Levina*, New York: Columbian University Press, 2004

Hanna, Robert. *Kant and Foundations of Analytic Philosophy*, Oxford: Clarendon Press, 2001, 196 https://www.academia.edu accessed June 27, 2021

Hamlyn David. *The theory of knowledge*, London: the Macmillan Press Ltd, 1970.

Heard, J., Scoular C., Duckworth, D., Ramalingam, D. and Teo I. "*Critical Thinking: Skill Development Framework"*, Australian council for Education Research, (2020), 6 https://research.acer.edu.au/ar_misc/41, Accessed July 3, 2021

Ige, J.A. *Boosting Resources Utilization in Biology Classroom*, 41 Annual Conference Proceeding of STAN, (2000), 141-145

Igbokwe, D.I. "*Dominant Factors and Error Types Inhibiting The Understanding of Mathematics"* Proceeding of the 41st Annual Conference of Science Teachers Association of Nigeria, (2000), 242-249

Ihejirika, C & Wolemonwu, V. The role of the subject in Rene Descartes theory of Knowledge, Filosofika: Univesity of Port Harcourt, Journal of Philosophy and Public Affriars, 1(1), 2015, 1-13, www.researchgate.net, Accessed on June 11, 2022

Irving, Copi and Cohen Carl. *Introduction to Logic*, 11th Edition, ISBN 81-203-2496, India: Prentice Hall Private Limited, 2005.

Jeremy, J. G., Craig, G., Frasher, J L., Berggren, M. F., and Wilbr, R. K. Mathematics, *Encyclopedia Britannica Inc*., 2019, https://www,britannica.com, Accessed January 7, 2021

Kanamori, Akihiro. "Cohen and set theory", Journal Article: The Bulletin of Symbolic Logic Volume 14 No3, (2008), 351-378.

Katz, Victor. *A history of mathematics*, ISBN 978-0-321-01618-8, Addison-Wesley, 1998.

Kong, S.C. "An experience of a three-year study on the development of critical thinking skills in flipped secondary classrooms with pedagogical and technological support". *Computers and Education Journal*, 89 (1), (2015).

Kulbir, Singh. *The Teachers of Mathematics*, New Delhi: Sternling Publishers Private Ltd (1), 2006

Kuiper, J.J. *The Role of Logic*, 2004, https://dspace.library.uu.nl, Accessed February 16, 2021

Maclean Ian. Rene Desacartes A Discourse on the Method, A New Translation, USA: Oxford University Press, 2006, 17-21.

Maduabum, M.A & Odili G.A. "Analysis of students' performance in general mathematics at SSCE level in Nigeria 1992-2002". *Journal Research for Teaching* 1(11) (1), (2006).

Manno, Michelle. *The Benefit of Puzzle in Early Childhood Development*, 2013, https://teach.com, Accessed July 11, 2021.

Mansoor, F & Pezeshki, M. "Manipulating critical thinking skills in test taking", *International Journal of Education* 4 (1), (2012).

Mendelman Lisa. "Critical thinking and reading". *Journal of Adolescent and Adult Literacy, 51(4*), (2007), 300-304, http://www.reading.org/General/Publications/Journals/jaal.aspx44, accessed August 26, 2021

Menkiti, Benedette.Uzoamaka. *Child Development and Education*; ISBN 978-36189-8-9, Nigeria: Chembus communication ventures, 2008.

Merriam Webster Dictionary

Nwachukwu, N. K. The effect of Study Habits on students achievement in chemistry in secondary schools, *The Nigeria Journal of Education* Vol 4, : Amalu, G. (1991) A Companion of male and female performance and a document in chemistry, [unpublished B.sc Edu] Enugu State University of Science and Technology. ISBN 978-2376-34-5, Nigeria: Blackblet Konzult Ltd, (2015).

Osarenren, N. & Asiedu A.S. "Counselling implication on the role of the new mathematics teacher in the teaching & learning mathematics". International Journal (1), (2007).

Paul Richard and Elder Linda, The Miniature Guide to Critical Thinking Concept and Tools, Dillon Beach: Foundation for Critical Thinking Press, 2010.
https://louisville.edu/ideastoaction/about/criticalthinking/framework Accessed July 27, 2021

Peschke, Karl. H. *Ethics; Moral Theology in the Light of Vatican II Special Moral Theology*, Revised Edition, India: Theological Publication, 1994.

Popper Karl. Conjectures and Refutation, London: Routledge, 1963, 26.

Popova Maria. *Rules of the Mind: Descartes' 12 Timeless Tenets of Critical Thinking*, https://www.brainpickings.org Accessed September 4, 2021

Rajendran, N.S. Teaching and acquiring higher order thinking skills: Theory and Practice Tanjong Malim, Perak: Penerbit Univesiti Pendidikan Sultan Idris, 2010.

Rameli Modh Rustam and Kosnin Azlina. "*The survey on mathematics achievement goal orientation among Malaysian students: Application of rasch measurement"*, India, 2017, https://www.researchgate.net, Accessed June 3, 2021

Rene Descartes. *Discourse on method and meditations on First Philosophy* 4th edition, Translated by Donald A., Cress Indianapolis/Cambridge: Hackett Publishing Company, 1998, 12

Rene Descartes. *The Method, Meditations and Philosophy*, Translated from the original texts, with a new introductory, essay, historical and critical by John Veitch, L.L.D, Washington: M. Walter Dunne Publisher, 1901, https://oll.libertyfund.org, Accessed 3/9/21

Rene Descartes. "*Rules for directing the mind" in the Philosophical works of Descartes* Vol.1(Trans.) Haldane E.D and Ross G.R.T, Cambridge: Murray Printing Co., 1911, 9.

Renati Des-Cartes. *Meditationes, De Prima Philosophia in Qua Dei Existentia et Animae Immortalitas Demonstratyr*, Parisii, Apud Michaelem Soly via Iacobea, fub figno Phoenicis M.DC. XLI cum Priuilegio Approbatione Dictorum. This file is of the 1911 edition of the Philosophical works of Descartes Translated by Elizabeth, S. Haldene, Cambridge: Cambridge University Press, Internet Encyclopedia of Philosophy 1996,1-32.

Reichenbach, Bruce. R. *Introduction to Critical Thinking*. Washington DC: American Psychological Association, 2001.

Rollings A. & Adams, Ernest and Rollings Andrew. *Fundamentals of Game Design (Game Design and Development Series)* USA: Prentice Hall Inc, 2006.

Russell, Bertrand. *History of Western Philosophy*, 2004.

Semil, R. "Enhancing thinking skills in the classroom". *Human and Social Sciences Journal* 1 (1). , (2006).

Sorell, Tom. *Descartes; A very Short Introduction*, New York: Oxford University Press 2000

Stanford Encyclopedia of Philosophy (2018). https://www.plato.stanford.edu, Accessed December 10, 2020

Stoylar, Abram. *Introduction to Elementary Mathematical Logic*, ISBN 978-0-486-64561-2, USA: Dover Publications, 1983, 3

Stumpf Samuel Enoch . Philosophy: *History and Problems*, New York: Mc-Graw Hill, 5th Edition, 1994, 227-228

Su Huang, Fang; Ricci, Fredrick., and Mnatsakania Mimikon. "Mathematical teaching strategy, pathway to critical thinking and Metacognition", *International Journal of Research in Education and Science* Volume II No.1, (2016), 190-200

Tarski Alfred. D*ecision Method for Elementary Algebra and Geometry*, California: Santa Monica, RAND Corporation, 1948.

Thomas, Keith. and Lok, Bearice. *Teaching Critical Thinking: An Operational Framework in M. Davies and R. Bernett (Eds)*. The Palgrave handbook of critical thinking in higher education, USA: Palgrave Macmillan, 2015, 93-105.

Tunca, N. "The regression level of constructivist learning environment characteristics on classroom environment characteristics supporting critical thinking". *Eurasian Journal of Educational Research*, 60, (2015).

Udigwomen, A.F. *Philosophy and The Rise of Modern Science*, ISBN 978-227-520-1, Nigeria: El-John publishers, 2011.

Unodiaku, S.S. "Development and validation of mathematics readiness test for senior secondary school students", *African Journal of Science, Technology and Mathematics Education* (AJSTME) 2 (1), (2012), 57-69

Unodiaku, S.S. "The influence of sex and ability level on students' mathematics readiness in Enugu State", *The Journal of education* Volume 4, ISBN 978-2376-34-5, Blackbelt Konzult Ltd , (2015),121-78-2376-34-5.

Vanguard Newspaper: WAEC release 2019 WASSCE result records 64% credit pass in English, Math, July 26, 2019.

Viviane Durand-Guerrier, Paolo B., Nadia D., Epp, S., and Tanguay, D. "*Examining the Role of Logic in Teaching Proof"*, Volume 15, New ICMI study series, (2012).

Wakefield, A. P. *Early Childhood Number Games, Teacher Reinvent Math Instruction Pre-kindergarten Through 3rd Grade*, Boston: Allyn and Bacon, 1998.

Watson Anne. and Mason John. *Questions and Prompt for Mathematical Thinking*,1998.

Westacott, E. *5 Good Reasons to Study Logic*, 2019, Retrieved from https://www.thoughtco.com/good-reason-to-study-logic-2670416, Accessed December 20, 2020

Yuliani, K. & Saragih, S. "The development of learning devices based guided discovery model to improve understanding concept and critical thinking mathematically ability of students at Islamic junior high school of Medan". *Journal of Education and Practice*, 6 (24), (2015).